Astrodynamics/Fundamental Laws

Astrodynamics

Newton's Laws of Motion

First Law (Law of Inertia)
: A body in motion tends to remain in motion in the same speed and direction unless acted upon by an external force. A body at rest tends to remain at rest unless acted upon by an external force.

Second Law (Law of Force)
: The rate of change of the speed of an object, its acceleration, is proportional to the force applied on that object, and occurs in the same direction as that force.

Third Law
: To every action there is always an equal but opposite reaction.

The second law is typically written by the equation $F = ma$, where a is the acceleration of the object, m is the mass of the object, and F is the amount of force applied. However, the second law also stipulates that the acceleration has a direction associated with it, and that direction is the same as the direction of the applied force. We must re-write this equation using vectors to account for the direction of the force and the acceleration. We can write this instead as $\boldsymbol{F} = m\boldsymbol{a}$, where the bold indicates the quantity has been vectorised. We will not define specifically what items these vectors contain, because as we will see shortly, it is very much dependent on the problem.

In space, there is never a single force acting on an object: gravitational pulls from all objects in the universe play some role. However, because the force that gravity exerts on an object decreases with the square of the distance, objects that are very far away produce negligible force, and can typically be ignored. However, consider a situation of a satellite orbiting around the earth: that satellite is being affected by gravity from the earth, from the moon, and to a lesser extent from the sun. We can sum these forces together to produce a more general form of the second law when the mass is constant. However, often times we need to transfer between our orbits or adjust our orbits due to additional forces. Newton's second law is based on the change in momentum which can be reduced to a simpler equation for a constant mass:

$$m\mathbf{a} = \sum \mathbf{F} = \sum \frac{d(m\mathbf{v})}{dt} = \sum(\mathbf{v}\frac{dm}{dt} + m\frac{d\mathbf{v}}{dt}) \approx \sum m\mathbf{v} \qquad \textbf{[Newton's Second Law]}$$

The third law also produces an interesting result, and one that is going to shape many calculations. In space travel there are two types of motion: powered and un-powered. In a powered device, such as a rocket, for the body to be "pushed" forward, it must push something backward with equal force. In the case of a rocket, burnt fuel is pushed out the back of the rocket, in order to move the body of the rocket

forward. As a result of the third law, any powered space craft will have *non-constant mass*, which will introduce additional complexity into our calculations. If we have two bodies, 1 and 2, and they are pushing against each other, we can say that the force exerted by body 1 on body 2 is equal in magnitude, but opposite in direction, from the force exerted by body 2 on body 1. In other words, we can say:

$$\mathbf{F}_{12} = -\mathbf{F}_{21}$$ **[Newton's Third Law]**

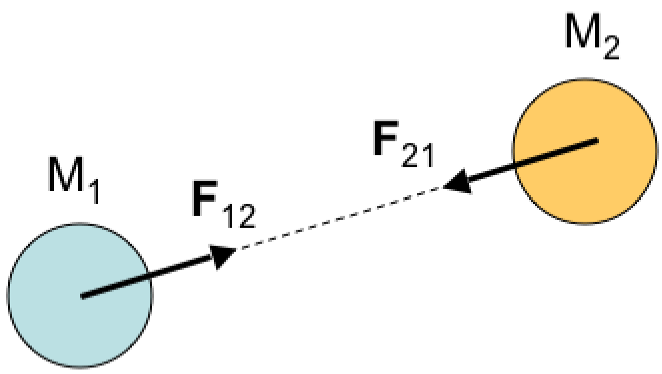
We will use these laws throughout the book.

Newton's Law of Universal Gravitation

Newton's law of gravity is another common law, and one that is of prime importance in the study of astrodynamics. Stated in its most simple form, the law of gravity is:

$$\mathbf{F}_g = -\frac{Gm_1m_2}{r^2}\hat{\mathbf{r}}$$ **[Law of Universal Gravitation]**

We know that the law of gravity is directional, that is that the force of gravity works to pull two bodies together, along the straight line between them. The image below shows two bodies, M_1 and M_2 being attracted to each other through the force of gravity acting along the line between their centres of gravity:



The force acting on M_1 is denoted as $\boldsymbol{F}_{12}$, and the force acting on M_2 is denoted as $\boldsymbol{F}_{21}$. We know from Newton's third law of motion that the two forces must be equal in magnitude, but opposite in direction:

$$\mathbf{F}_{12} = -\mathbf{F}_{21}$$

Because the direction of the force is important, we vectorize the law of universal gravitation equation as well:

$$\mathbf{F}_g = -\frac{Gm_1m_2}{r^2}\frac{\mathbf{r}}{r}$$

Where **r** is the distance vector between the two bodies, and *r* is the scalar distance between the two. $\mathbf{F}_g$ is the force vector of gravity.

Gravitational Constant

The big G is called the **gravitational constant**. It is equal to:

$$G = 6.67428 \times 10^{-11}\ \mathrm{m}^3\ \mathrm{kg}^{-1}\ \mathrm{s}^{-2}$$

Significance

Although Newton's law of gravity is not exact as compared to relativity, it is often used as it provided great results for most cases. One simple consequence of this law is the idea of an escape velocity, which will shortly be derived. The concept of the escape velocity signifies that it's not a force, nor height that needs to be achieved to escape the pull (gravitational influence) of a planet or orbiting body.

There are a few ways to derive the escape velocity however the most general way is by using Newton's Law of Gravity and basic calculus.

Let assume that a mass, m_2, is traveling radially away form the planet of mass, m_1. Therefore, the equations can be expressed in 1D.

$$F_g = -\frac{Gm_1m_2}{r^2}$$

If the mass moves away from the main body by a small distance dr and recalling the fact that work energy is a force multiplied by the colinear distance moved then the small amount of work needed to be done to move the mass is:

$$dW = Fdr = -\frac{Gm_1m_2}{r^2}dr$$

The force of gravity extends from here to anywhere else in the Universe the only possible way for the mass to not be acted on by the force of gravity is for it to be infinitaly far away. Let say the mass is moved from a distance a from the planet. Then the work needed to move it from a to infinity is:

$$W = -\int_a^\infty \frac{Gm_1m_2}{r^2}dr = -\frac{Gm_1m_2}{a}$$

As stated above it is the work need to move the mass; thus, for the mass to move this distance it needs to start with at least the same amount of kinetic energy.

$$\frac{1}{2}m_2v_0^2 = \frac{Gm_1m_2}{a}$$

$$v_0 = \sqrt{\frac{2Gm_1}{a}}$$

Therefore, the escape velocity is equal to v_0.

Kepler's Laws of Planetary Motion

First Law

The orbit of a planet is an ellipse with the sun at one of the foci of that ellipse.

Second Law

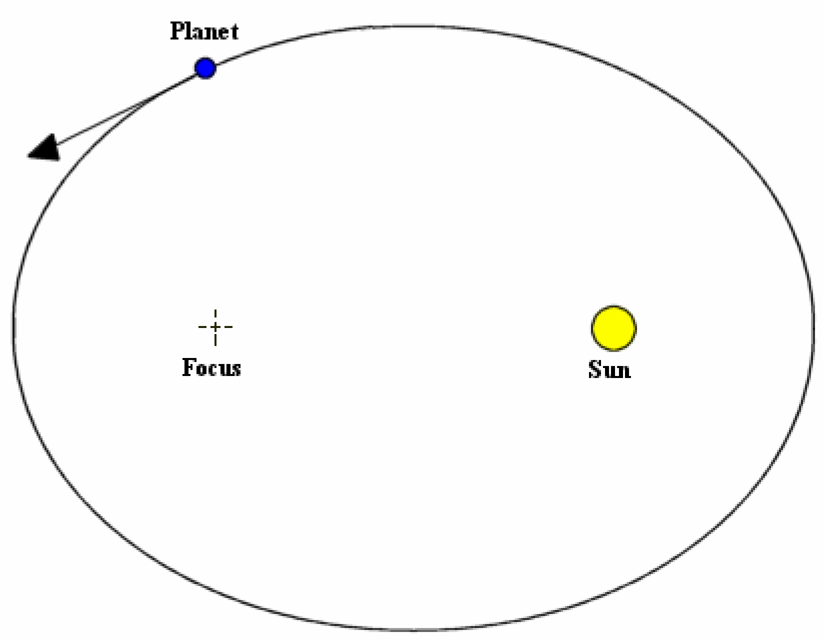
The line joining the planet to the sun sweeps out equal areas in equal periods of time.

Third Law

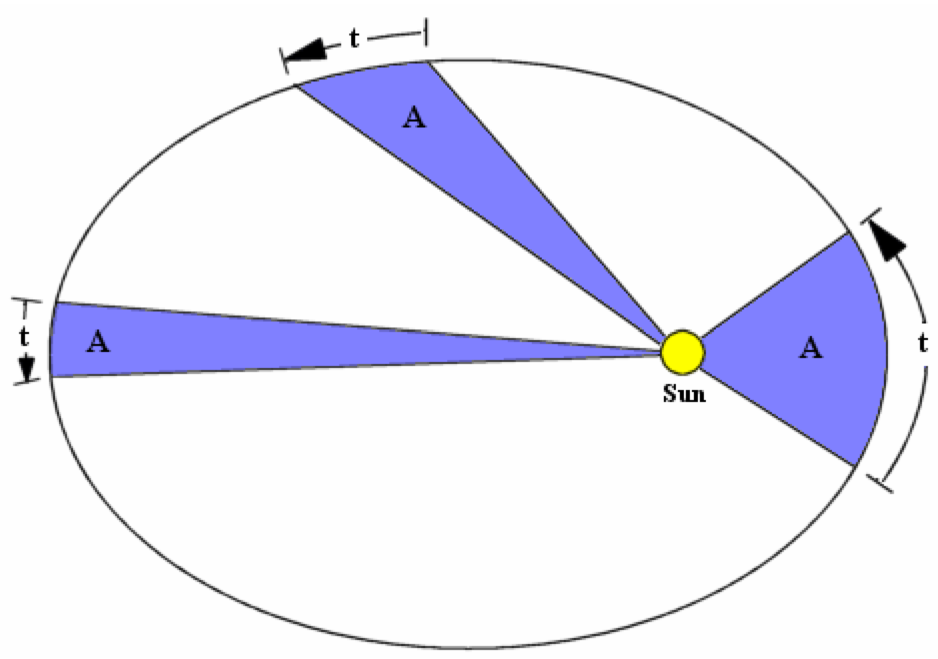
The square of the planet's period is proportional to the cube of its average distance from the sun.

As we remember from algebra, an ellipse is a conic section formed from two foci, where the sum of the distances from the ellipse to each focus is equal. The sun is at one focus, or any body that is being orbited, and the other focus is an imaginary point in space. It is important to mention that Kepler's law also extent to a circle which is a special case of an ellipse.

This picture is an orbit in the form of an ellipse with the sun at one of the foci. We call the focus that contains the central body the *prime focus*, and the other focus, which has no physical significance as the *secondary focus*.



This picture is a demonstration of Kepler's second law:



As we can see, for the line joining the planet (dashed lines) to sweep out triangular sections of equal areas, it must move slower as the planet gets further from the sun. When the planet is closer to the sun, it must be traveling faster in order to sweep out equal areas in equal time. This follows exactly as expected from Newton's laws of motion and gravity where the force is greater near the *prime focus*, thus the acceleration is larger and hence the velocity.

For a similar example, consider a bouncing ball. The ball is moving slowest at the top of its arc, and continues to gain speed as it travels downward towards the Earth. As the Earth moves away from the sun, gravity decelerates the planet. When the planet is moving towards the sun, gravity is accelerating the planet.

Kepler's third law can be easily illustrated by comparing the average distances of solar system bodies and their orbital periods to Earth's. Expressed in astronomical units and years the orbital period is the semi-major axis raised to 3/2 power.

Object	Semi-major axis (AU)	Orbital period (years)
Mercury	0.39	0.24
Venus	0.72	0.62
Earth	1.00	1.00
Mars	1.67	1.88
Ceres	2.77	4.60
Jupiter	5.20	11.86
Saturn	9.58	29.46
Uranus	19.23	84.32
Neptune	30.10	164.79
Pluto	39.26	247.68
Eris	68.01	560.9
Sedna	518.57	≈11400

Derivation of Kepler's Laws of Planetary Motion